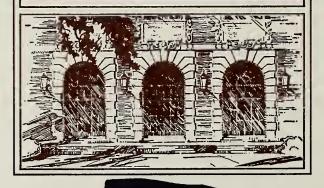
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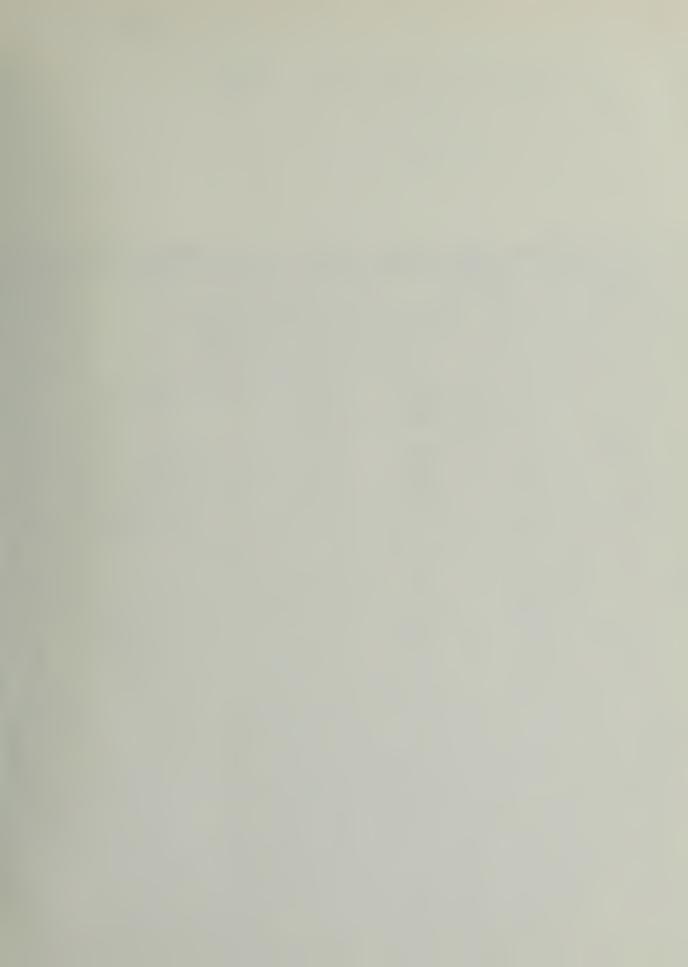
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## SPERNER'S THEOREM ON MAXIMAL-SIZED ANTICHAINS AND ITS GENERALIZATION

by

C. L. Liu

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# SPERNER'S THEOREM ON MAXIMAL-SIZED ANTICHAINS AND ITS GENERALIZATION

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#### 1. Introduction

Let S be a set of n elements. Let  $F_0 = \{A_1, A_2, \dots, A_i, \dots\}$  be a family of subsets of S such that no two subsets  $A_i$  and  $A_j$  in  $F_0$  possess the property  $A_i \supseteq A_j$ . Sperner [1] proved that

$$|F_0| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}^{\dagger}$$

Sperner's result was generalized by Erdös [2] who showed that the size of a family of subsets of S such that no  $\ell+1$  subsets  $A_1$ ,  $A_2$ , ...,  $A_1$  in the family form a chain  $A_1 \supseteq A_1 \supseteq \ldots \supseteq A_1$  is upperbounded by the sum of the  $\ell$  largest binomial coefficients of order n. Kleitman [3] and Katona [4] improved Sperner's result in the following way:

Let  $S_1 \cup S_2 = S$  and  $S_1 \cap S_2 = \emptyset$  be a partition of S. Let  $F_0 = \{A_1, A_2, \ldots, A_1, \ldots\}$  be a family of subsets of S such that no two subsets  $A_1$  and  $A_2$  in  $F_0$  possess the properties:

(i) 
$$(A_i \cap S_1) = (A_j \cap S_1)$$
 and  $(A_i \cap S_2) \supset (A_j \cap S_2)$ ; or

(ii) 
$$(A_i \cap S_1) \supset (A_j \cap S_1)$$
 and  $(A_i \cap S_2) = (A_j \cap S_2)$ 

Then

$$|\mathbf{F}_{0}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

t | k | denotes the largest integer not larger than k.



factors of the integer. We use the notation  $S_m^i(N)$  to denote the number of divisors of N that are of degree i, where m is the degree of N. We follow the convention that  $S_m^i(N) = 1$  for i = 0 and  $S_m^i(N) = 0$  for i < 0 and i > m. Let  $F_0 = \{a_1, a_2, \dots, a_i, \dots\}$  be a set of divisors of N such that no two divisors  $a_i$  and  $a_j$  in F. possess the property  $a_i \mid a_j$ . According to DeBruijn, Tengbergen, and Kruyswijk,

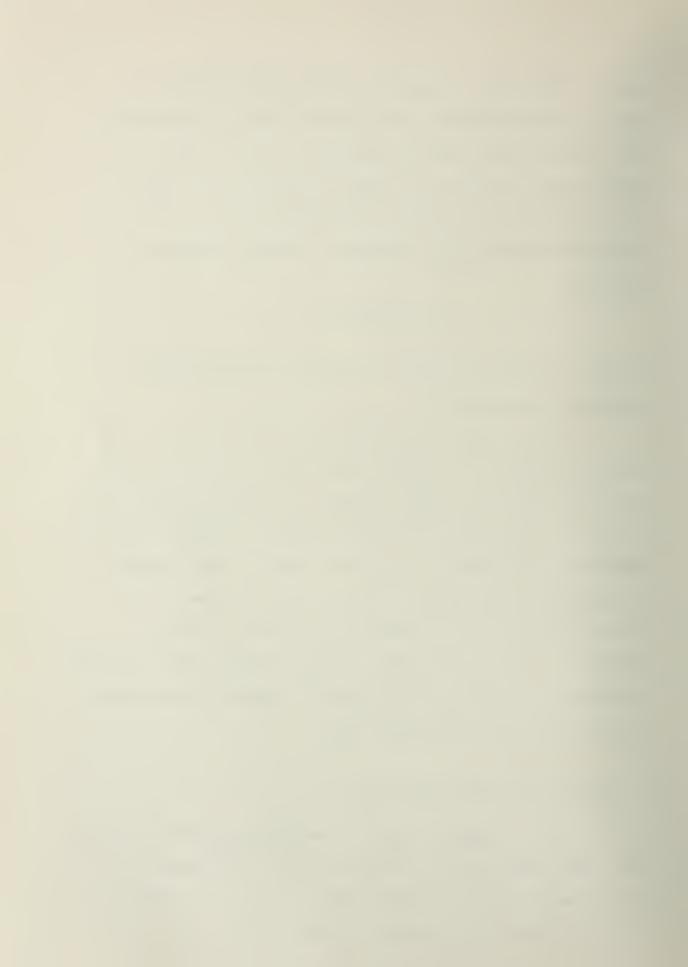
$$|F_0| \le S_m^{\lfloor m/2 \rfloor}(N)$$

Recently, the result of DeBruijn, Tengbergen, and Kruyswijk was generalized by Schönheim [6].

Katona [7] obtained a general result that includes all the results mentioned above as special cases. In this paper, we present a result that is similar to that of Katona's. Our result is more general in that it is applicable to the direct products of arbitrary partially ordered sets, yet Katona's result is restricted to direct products of "symmetrical chain graphs". For the lattice of subsets of a set and the lattice of divisors of an integer, we obtain simple proofs of many of the known results, and also are able to sharpen some of them. Our main result is Theorem 3. However, for exposition purpose we show first some special cases.

## 2. Product of partially ordered sets

Let  $P = \{p_1, p_2, \dots, p_i, \dots\}$  be an arbitrary partially ordered set. Let  $L_h$  denote the set of integers  $\{0,1,2,\dots,h\}$  ordered by the "larger than or equal to" relation. Let  $R = P \times L_h$ . The elements in R shall be denoted  $(p_i,j)$ , where  $p_i \in P$  and  $j \in \{0,1,2,\dots,h\}$ . Let



 $T \subseteq R$ . An element  $(p_i,j)$  in T is said to be an <u>unobstructed</u> element in T if there is no other element  $(p_i',j')$  in T such that  $p_i > p_i'$  and j = j' or  $p_i = p_i'$  and j > j'. We also introduce the notation, for a fixed j,

$$T(j) = \{ (p_{i}, j) \mid (p_{i}, j) \in T \}$$

$$T(\overline{j}) = \{ (p_{i}, j') \mid (p_{i}, j') \in T, \quad j' > j \}$$

$$\overline{T}(\overline{j}) = \{ (p_{i}, j) \mid (p_{i}, j') \in T, \quad j' > j \}$$

We shall use  $P[\ell]$  to denote a maximal-sized subset of P such that no  $\ell+1$  elements in the subset form a chain of length  $\ell+1$ . It will be understood that  $P[\ell] = \emptyset$  for  $\ell \leq 0$ .

Theorem 1: Let  $R = P \times L_h$ . Let  $F_O$  be a subset of R such that no  $\ell$ +1 elements in  $F_O$  form a chain of length  $\ell$ +1. Then for  $\ell$  > h

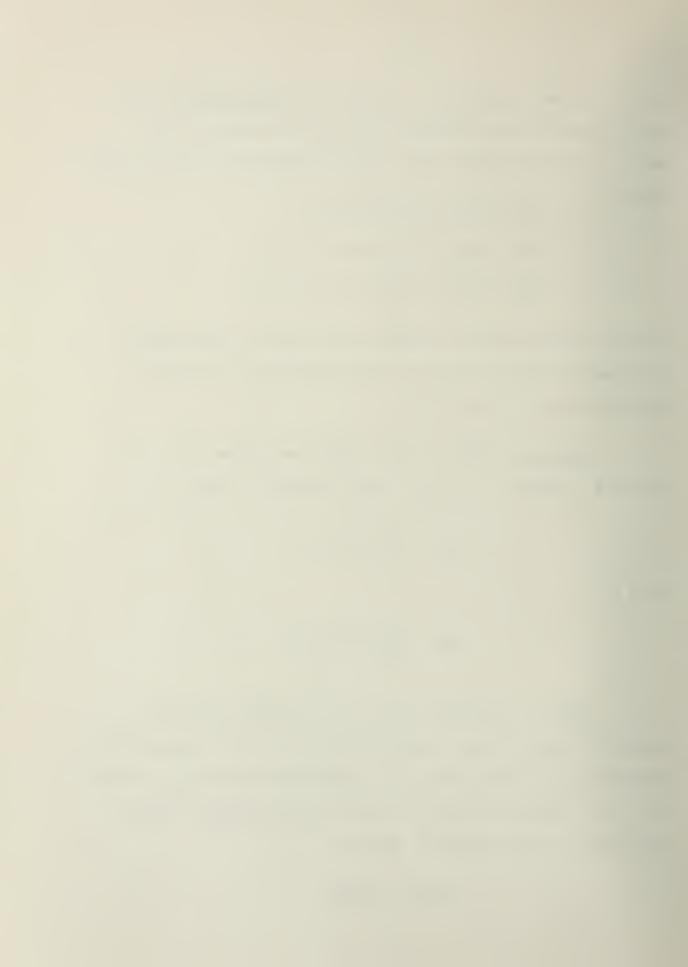
$$|F_0| \le \sum_{t=0}^{h} |P[\ell+h-2t]|$$

for  $\ell \leq h$ 

$$|F_0| \le \sum_{t=0}^{\ell-1} |P[\ell+h-2t]|$$

<u>Proof:</u> We prove the case  $\ell > h$ . The case  $\ell \le h$  can be proved in a similar manner. Let  $M_O$  denote the set of all unobstructed elements in  $F_O$ . We note that  $(p_i,j) \in M_O(\overline{O})$  implies that  $(p_i,j') \notin M_O(\overline{O})$  for  $j' \ne j$ . Therefore, there is a one-to-one correspondence between the elements in  $M_O(\overline{O})$  and  $\overline{M}_O(\overline{O})$ . That is,

$$\left| \mathbf{M}_{\mathbf{O}}(\overline{\mathbf{O}}) \right| = \left| \overline{\mathbf{M}}_{\mathbf{O}}(\overline{\mathbf{O}}) \right|$$



We also note that  $(p_i,j) \in M_O(\overline{O})$  implies that  $(p_i,0) \notin F_O$ . That is, the sets  $\overline{M}_O(\overline{O})$  and  $F_O(O)$  are disjoint. Consequently, we have

$$\left| \mathbf{F}_{\mathbf{O}} \right| = \left| \mathbf{F}_{\mathbf{O}}(0) \cup \overline{\mathbf{M}}_{\mathbf{O}}(\overline{0}) \right| + \left| \mathbf{F}_{\mathbf{O}} - \mathbf{F}_{\mathbf{O}}(0) - \mathbf{M}_{\mathbf{O}}(\overline{0}) \right|$$

Recursively, for t = 1, 2, ..., h let  $F_t$  denote the set  $F_{t-1} - F_{t-1}(t-1) - M_{t-1}(\overline{t-1}) \text{ and } M_t \text{ denote the set of unobstructed}$  elements in  $F_t$ . For t = 1, 2, ..., h-1, because

(i) 
$$|M_{t}(\overline{t})| = |\overline{M}_{t}(\overline{t})|$$

(ii) the sets  $\overline{M}_{t}(\overline{t})$  and  $F_{t}(t)$  are disjoint

we have

$$|\mathbf{F}_{\mathsf{t}}| = |\mathbf{F}_{\mathsf{t}}(\mathsf{t}) \cup \overline{\mathbf{M}}_{\mathsf{t}}(\overline{\mathsf{t}})| + |\mathbf{F}_{\mathsf{t}} - \mathbf{F}_{\mathsf{t}}(\mathsf{t}) - \mathbf{M}_{\mathsf{t}}(\overline{\mathsf{t}})|$$

It follows that

$$|\mathbf{F}_{0}| = \sum_{t=0}^{h-1} |\mathbf{F}_{t}(t) \cup \overline{\mathbf{M}}_{t}(\overline{t})| + |\mathbf{F}_{h}|$$

Since the set  $F_0(0)$  does not contain chains of length larger than  $\ell$  and the set  $\overline{M}_0(\overline{O})$  does not contain chains of length larger than h, we have

$$|F_{O}(0) \cup \overline{M}_{O}(\overline{0})| \leq |P[\ell+h]|$$

Similarly, for t = 1,2,...,h-1, since the set  $F_t(t)$  does not contain chains of length larger than  $\ell$ -t and the set  $\overline{M}_t(\overline{t})$  does not contain

Because every element of  $F_t(t)$  dominates a chain of length t.



chains of length larger than h-t<sup>†</sup>, we have

$$|F_{t}(t) \cup \overline{M}_{t}(\overline{t})| \le |P[\ell+h-2t]|$$

Finally, because  $F_h$  does not contain chains of length larger than  $\ell$ -h, we have

$$|F_h| \le |P[l-h]|$$

We thus obtain

$$|F_0| \le \sum_{t=0}^{h} |P[\ell+h-2t]|$$

Corollary 1.1: Let  $R = P \times L_1$ . Let  $F_0$  be a subset of R such that no  $\ell$ +1 elements in  $F_0$  form a chain of length  $\ell$ +1. Then

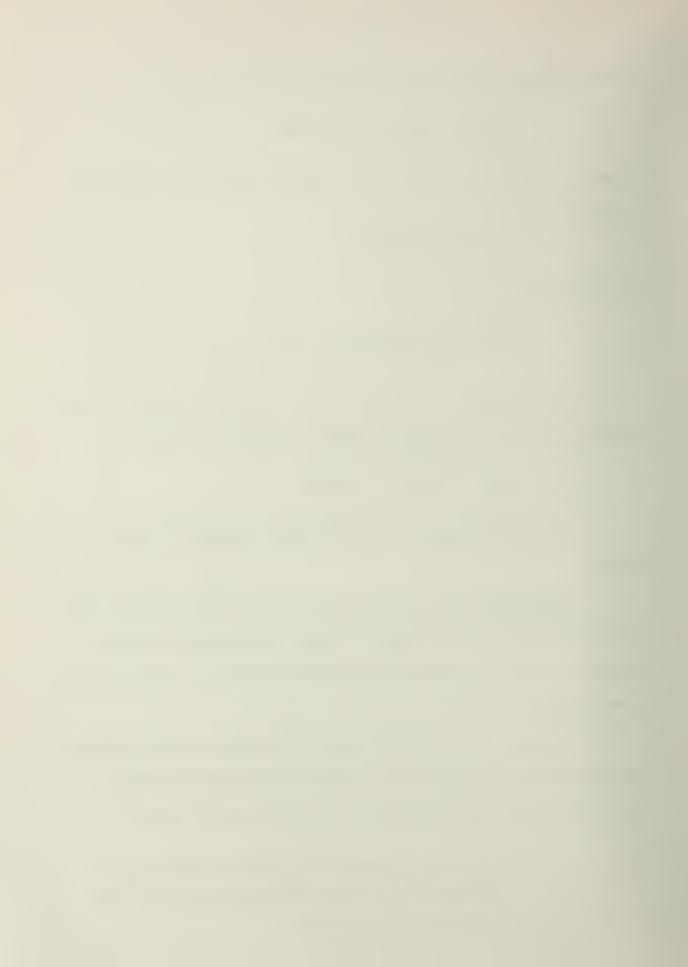
$$|F_0| \le |P[\ell+1]| + |P[\ell-1]|$$

From Corollary 1.1 we obtain Erdös' extension of Sperner's result:

Corollary 1.2: In the lattice of subsets of a finite set of n elements, the size of a family of subsets that does not contain a chain of length 1+1 is upperbounded by the sum of the 1 largest binomial coefficients of order n.

<u>Proof:</u> Let  $R = (L_1)^{n-1} \times L_1$ . We can then prove the corollary by induction on n, using the fact that the sum of the  $\ell$ +1 largest binomial coefficients of order n-1 and the  $\ell$ -1 largest binomial

If  $(p_i,j)$  and  $(p_i',j)$  are two elements in  $M_t(\overline{t})$ , then they must be incomparable. Consequently, the corresponding elements  $(p_i,t)$  and  $(p_i',t)$  in  $\overline{M}_t(\overline{t})$  must also be incomparable.



coefficients of order n-l is equal to the sum of the  $\ell$  largest binomial coefficients of order n.  $\dagger$ 

Similarly, we can obtain Schönheim's extension of DeBruijn,
Tengbergen, and Kruyswijk's result.

Corollary 1.3: In the lattice of divisors of an integer N, the size of a family of divisors that does not contain a chain of length  $\ell$ +1 is upperbounded by the sum of the  $\ell$  largest values of  $S_m^i(N)$ , where m is the degree of N.

Proof: Let 
$$N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{n-1}^{\alpha_{n-1}} p_n^{\alpha_n}$$
. Let  $R = (L_{\alpha_1} \times L_{\alpha_2} \times L_{\alpha_$ 

...  $\times$  L $_{\alpha_{n-1}}$ )  $\times$  L $_{\alpha_n}$ . We can prove the corollary by induction on n, using the result that the sum of the  $\ell$  largest values of  $S^1_{m+\alpha}(Np^{\alpha})$  is equal to

$$\sum_{k=0}^{\alpha} [\text{sum of the } \ell + \alpha - 2t \text{ values of } S_{m}^{i}(N)]$$

for  $\ell > \alpha$ , and is equal to

$$\sum_{t=0}^{\ell-1} [\text{sum of the } \ell+\alpha-2t \text{ values of } S_m^{i}(N)]$$

for  $l \leq \alpha$ . This result can be obtained routinely from the facts:

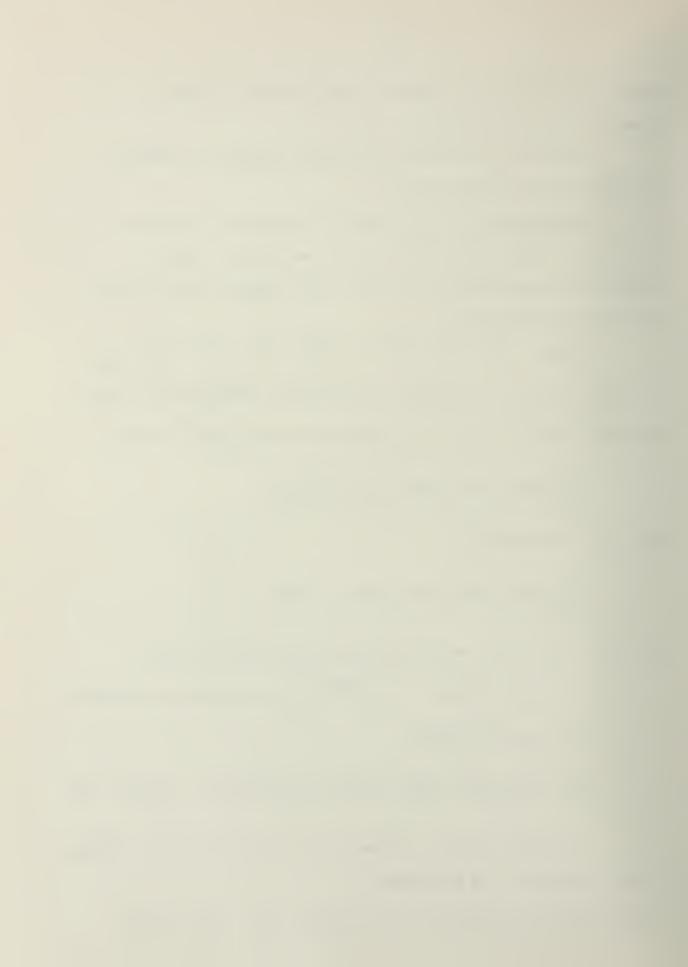
(i) 
$$S_m^1(N)$$
,  $S_m^2(N)$ , ...,  $S_m^{\lfloor m/2 \rfloor}(N)$  is a non-decreasing sequence

(ii) 
$$S_m^j(N) = S_m^{m-j}(N)$$

(iii) 
$$S_{m+\alpha}^{j}(Np^{\alpha}) = S_{m}^{j}(N) + S_{m}^{j-1}(N) + S_{m}^{j-2}(N) + \dots + S_{m}^{j-\alpha}(N)$$

Examining the proof of Theorem 1, we realize that the condition stated in Theorem 1 can be weakened:

This result comes directly from the relation:  $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$ 



Theorem 2: Let  $R = P \times L_h$ . Let  $F_0$  be a subset of R such that there are no  $\ell$ +1 elements  $(p_i, j_1), (p_i, j_2), \ldots, (p_i, j_\ell),$   $(p_i, j_{\ell+1})$  in  $F_0$  possessing the properties: for some  $t, 0 \le t \le h$ ,

(i) 
$$p_{i_1} > p_{i_2} > \dots > p_{i_{\ell-t+1}}$$
 and  $j_1 = j_2 = \dots = j_{\ell-t+1} = t;$ 

(ii) for 
$$k = \ell-t+1$$
, ...,  $\ell$ , either  $p_{i_k} > p_{i_{k+1}}$  and  $j_k = j_{k+1}$ , or  $p_{i_k} = p_{i_{k+1}}$  and  $j_k > j_{k+1}$ .

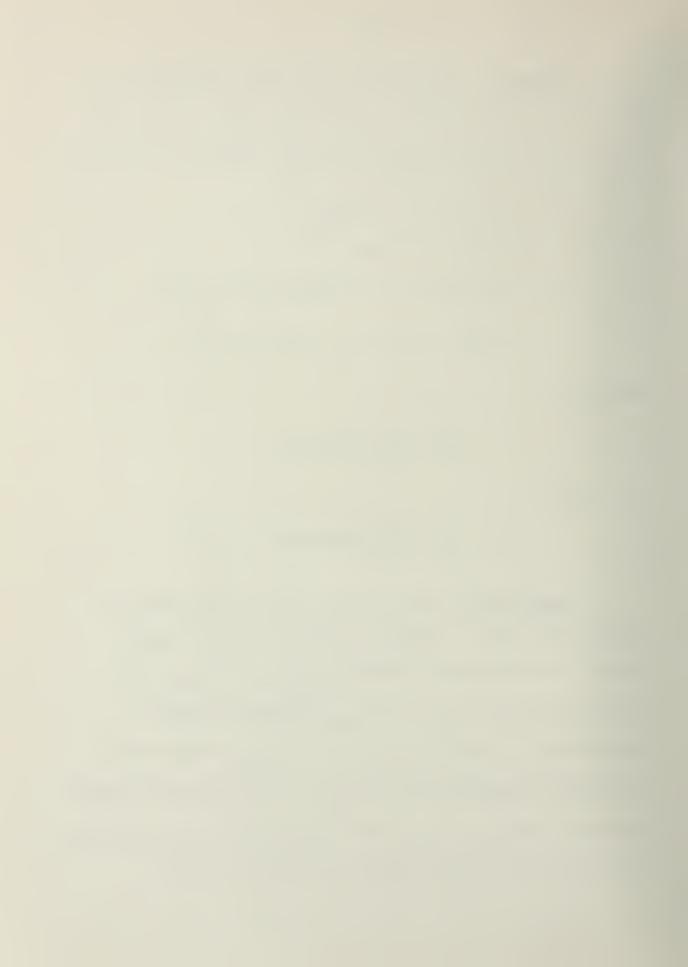
Then for  $\ell > h$ 

$$|F_0| \le \sum_{t=0}^{h} |P[\ell+h-2t]|$$

for  $l \leq h$ 

$$|F_0| \le \sum_{t=0}^{\ell-1} |P[\ell+h-2t]|$$

Corollary 2.1: Let  $F_0 = \{A_1, A_2, \dots, A_i, \dots\}$  be a family of subsets in the lattice of subsets of a finite set S of n elements. Suppose that there are no  $\ell+1$  subsets  $A_{i_1}, A_{i_2}, \dots, A_{i_{\ell+1}}$  in  $F_0$  such that either  $A_{i_1} \supset A_{i_2} \supset \dots \supset A_{i_{\ell+1}}$  and they all contain a certain element a,  $a \in S$ ; or  $A_{i_1} \supset A_{i_2} \supset \dots \supset A_{i_{\ell+1}}$  and they all do not contain the element a; or  $A_{i_1} \supset A_{i_2} \supset \dots \supset A_{i_{\ell+1}}$  and they all contain the element a and  $A_{i_{\ell+1}} = A_{i_{\ell}} - \{a\}$ . Then the size of  $F_0$  is upperbounded by the sum of the  $\ell$  largest binomial coefficients of order n.



Corollary 2.1 sharpens Frdös' result. If we set  $\ell=1$  in Corollary 2.1 we obtain a special case of Kleitman and Katona's result. Specifically, this case corresponds to partitioning S into S<sub>1</sub> and S<sub>2</sub> such that S<sub>1</sub> = {a} and S<sub>2</sub> = S - {a}. We shall present a more general result in Corollary 3.1.

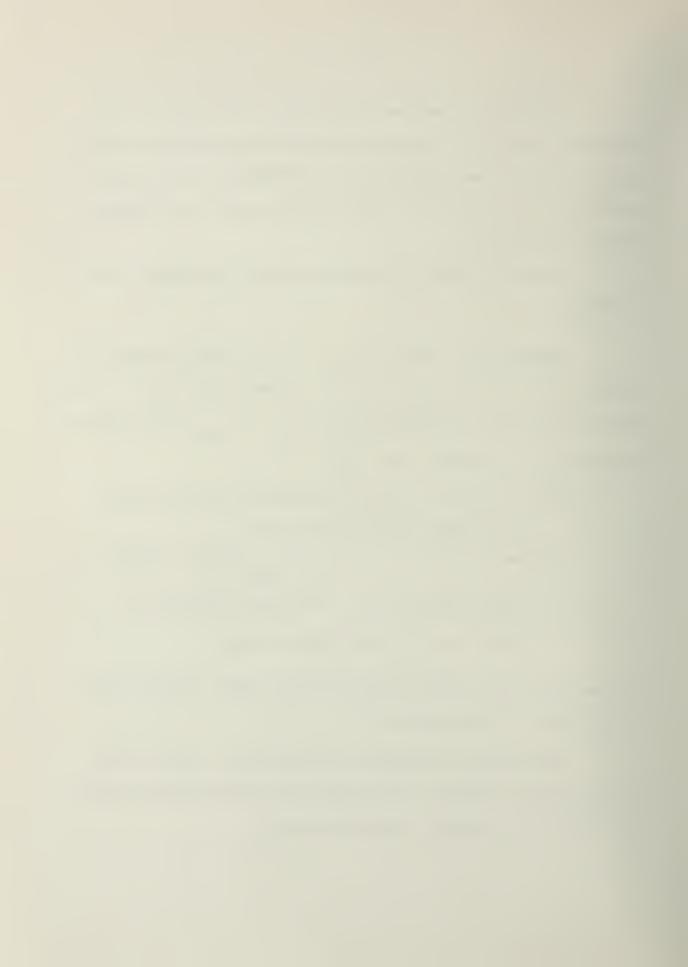
Similarly, Schönheim's result as stated in Corollary 1.3 can be sharpened:

Corollary 2.2: Let  $F_0 = \{a_1, a_2, \dots, a_i, \dots\}$  be a collection of integers in the lattices of divisors of an integer  $N = p_1^{\alpha_1} \quad p_2^{\alpha_2} \quad \dots \quad p_n^{\alpha_n}$  such that there are no  $\ell$ +1 divisors  $a_1$ ,  $a_1$ , ...,  $a_i$  in  $F_0$  possessing the properties: for some t,  $0 \le t \le \alpha_n$ ,

- (i)  $a_1 \div a_2 \div \cdots \div a_{\ell-t+1}^{\dagger}$  and they all contain exactly the  $t^{th}$  power of  $p_n$  for  $0 \le t \le \alpha_n$ ;
- (ii) for  $k = \ell t + 1$ , ...,  $\ell$ ,  $a_i \div a_{k+1}$  and the quotient of  $a_i$  divided by  $a_i$  either does not contain a power of  $p_n$  or is only a power of  $p_n$ .

Then the size of  $F_0$  is upperbounded by the sum of the  $\ell$  largest values of  $S_m^i(N)$  where m is the degree of N.

Let P and Q be two partially ordered sets. Let  $R = P \times Q$ . Suppose that the elements in Q are partitioned into d disjoint chains. The write  $a_i \div a_j$  to mean  $a_i$  is divisible by  $a_j$ .



Let h<sub>1</sub>, h<sub>2</sub>,..., h<sub>d</sub> denote the lengths of these chains. A direct consequence of Theorem 2 is

Theorem 3: Let  $F_0$  be a subset of R such that no  $\ell$ +1 elements in  $F_0$ ,  $(p_i, q_j)$ ,  $(p_i, q_j)$ , ...,  $(p_i, q_j)$  possess the properties:

(i) 
$$(p_{i_1}, q_{j_1}) > (p_{i_2}, q_{j_2}) > \dots > (p_{i_{\ell+1}}, q_{j_{\ell+1}});$$

- (ii)  $q_j$ ,  $q_j$ , ...,  $q_j$  are in the same chain in the partition of Q into disjoint chains;
- (iii) if in the chain containing  $q_{j_1}$ ,  $q_{j_1}$  is the t<sup>th</sup> element,  $q_{j_1} = q_{j_2} = \cdots = q_{j_{\ell-t+1}}$ ;
  - (iv) for  $k = \ell$ -t+1, ...,  $\ell$ , either  $p_{i_k} > p_{i_{k+1}}$  and  $q_{j_k} = q_{j_{k+1}}$ , or  $p_{i_k} = p_{i_{k+1}}$  and  $q_{j_k} > q_{j_{k+1}}$ .

Then

$$|\mathbf{F}_{0}| \leq \sum_{i=1}^{d} \mathbf{Z}_{i}$$

where for  $l > h_i$ 

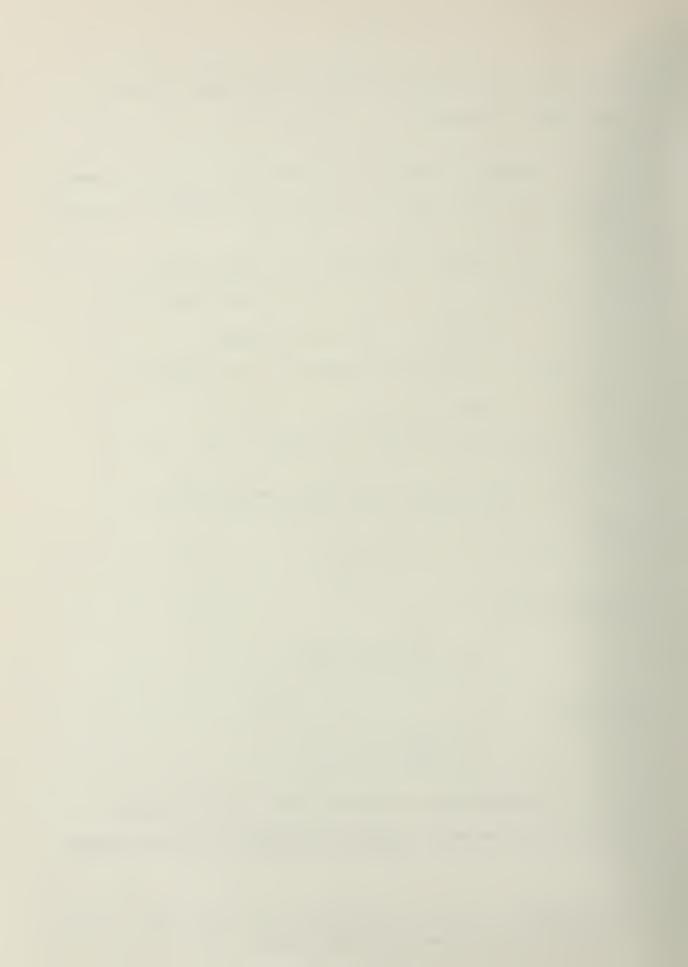
$$Z_{i} = \sum_{t=0}^{h_{i}} |P[\ell+h_{i}-2t]|$$

for  $\ell \leq h_i$ 

$$Z_{i} = \sum_{t=0}^{\ell-1} |P[\ell+h_{i}-2t]|$$

In order to apply Theorems 3 to the lattice of subsets of a finite set, we define a canonical partition of  $(L_1)^n$  into disjoint

The elements in a chain are labelled as the Oth, 1st, 2nd, ..., tth, ... elements, starting at the bottom of the chain.



chains trecursively as follows:

- (i) L is partitioned into a chain of length 2.
- (ii) Let  $p_1 > p_1 > \dots > p_m$  be a chain in a canonical partition of  $(L_1)^{n-1}$ . Then

$$(p_{i_1},1) > (p_{i_1},0) > (p_{i_2},0) > \dots > (p_{i_{m-1}},0) > (p_{i_m},0)$$

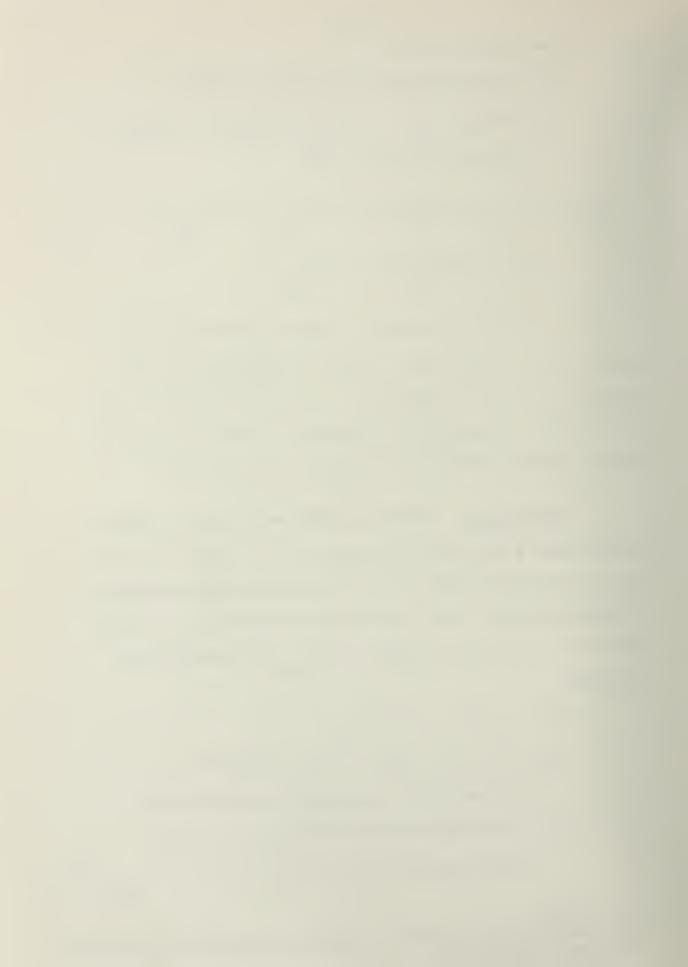
$$(p_{i_2},1) > (p_{i_3},1) > \dots > (p_{i_m},1)$$

will be two chains in a canonical partition of  $(L_1)^n$ . Let  $P = (L_1)^k$ ,  $Q = (L_1)^n$ , and  $R = P \times Q$ . By induction on n, we can immediately show that corresponding to a canonical partition of Q into disjoint chains, the sum  $\sum_{i=1}^d Z_i$  in Theorem 3 is equal to the sum of the largest  $\ell$  binomial coefficients of order k+n. We thus obtain:

Corollary 3.1: Let S be a finite set of size n. Suppose S is partitioned into  $S_1$  and  $S_2$  such that  $S_1 \cup S_2 = S$  and  $S_1 \cap S_2 = \emptyset$ . Let us partition the subsets of  $S_2$  into disjoint chains according to a canonical partition. Let F be a family of subsets of S such that there are no  $\ell+1$  subsets  $A_1$ ,  $A_1$ , ...,  $A_1$  in F possessing the properties:

- (i)  $A_{i_1} \supset A_{i_2} \supset \dots \supset A_{i_{\ell+1}}$
- (ii)  $A_{i_1} \cap S_1$ ,  $A_{i_2} \cap S_2$ , ...,  $A_{i_{\ell+1}} \cap S_2$  are in the same chain in the canonical partition of  $Q_s$
- (iii) if in the chain containing  $A_{i_1} \cap S_2$ ,  $A_{i_1} \cap S_2$  is the t<sup>th</sup> element,  $A_{i_1} \cap S_2 = A_{i_2} \cap S_2 = \cdots = A_{i_{\ell-t+1}} \cap S_2$

These indeed are what are known as symmetric chains defined by DeBruijn, Tengbergen, and Kruyswijk.



(iv) for 
$$k = l-t+1$$
, ...,  $l$ , either
$$A_{i_k} \cap S_1 \supset A_{i_{k+1}} \cap S_1 \text{ and } A_{i_k} \cap S_2 = A_{i_{k+1}} \cap S_2, \text{ or}$$

$$A_{i_k} \cap S_1 = A_{i_{k+1}} \cap S_1 \text{ and } A_{i_k} \cap S_2 \supset A_{i_{k+1}} \cap S_2.$$

Then the size of  $F_0$  is upperbounded by the sum of the  $\ell$  largest binomial coefficients of order n.

For  $\ell=1$ , Corollary 3.1 is reduced to: Let  $F_0$  be a family of subsets of S such that no two subsets  $A_i$  and  $A_j$  in  $F_0$  possess the properties

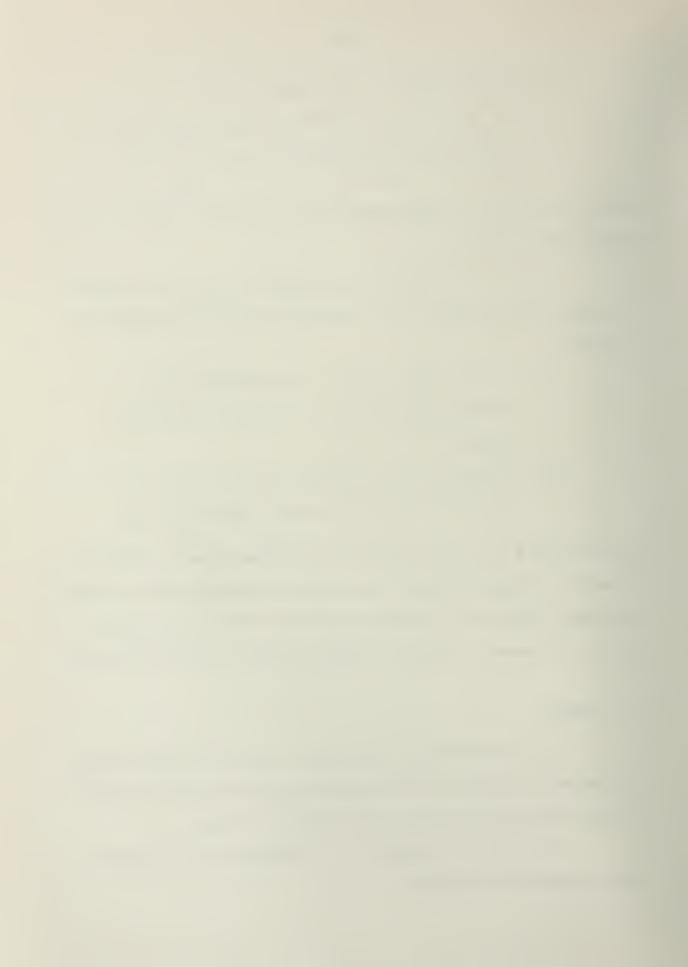
- (i)  $A_i \cap S_2$  and  $A_j \cap S_2$  are in the same chain in a canonical partition of the lattice of subsets of  $S_2$ ; and
- (ii) either  $(A_i \cap S_1) \supset (A_j \cap S_1)$  and  $(A_i \cap S_2) = (A_j \cap S_2)$ , or  $(A_i \cap S_1) = (A_j \cap S_1)$  and  $(A_i \cap S_2) \supset (A_j \cap S_2)$ .

Then the size of  $F_0$  is upperbounded by the largest binomial coefficient of order n. Clearly, Theorem 3 also leads to a generalization of DeBruijn, Tengbergen, and Kruyswijk's result in the same sense as Corollary 3.1 generalizes Sperner's result. We shall leave the details to the reader.

#### 3. Remarks

It is interesting to note that our approach is quite similar to Kleitman's approach [8] in solving Littlewood and Offord's problem on the distributions of linear combinations of vectors.

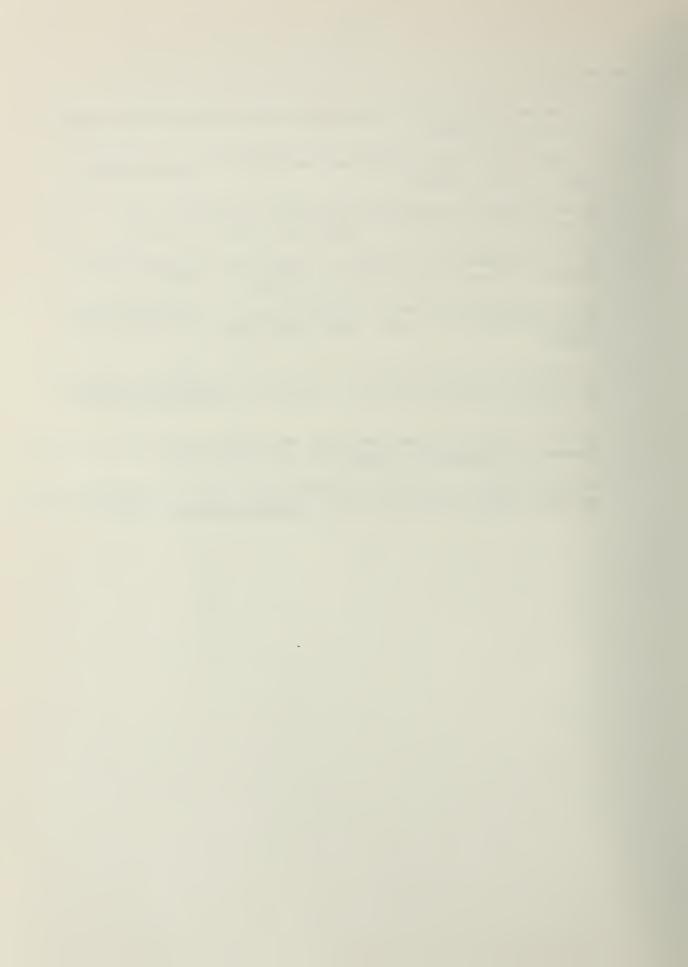
The author is indebted to D. J. Kleitman and C. Greene for their suggestions and comments.



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